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ABSTRACT

Eight sixth-grade students received individualized instruction on the addition and subtraction of fractions in a one-to-one setting for 6 weeks. Instruction was specifically designed to build upon the student's prior knowledge of fraction ideas and to emphasize estimation. It was determined that all students possessed a rich store of prior knowledge about estimation that was focused on parts of wholes in real world situations. On their own initiative, students related fraction symbols and procedures to prior knowledge about estimation in ways that were meaningful to them and accurately estimated sums and differences prior to learning computational algorithms. (PK)

USING ESTIMATION TO LEARN FRACTIONS WITH UNDERSTANDING

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In the last few years, estimation has been recognized as a basic skill that underlies students' understanding of mathematical symbols and procedures (National Council of Supervisors of Mathematics, 1977; National Council of Teachers of Mathematics (NCTM), 1980, 1987; Romberg, Bell, Senese, Willoughby, & Smith, 1984). Frequently estimation is portrayed as an algorithmic process based upon prior knowledge of formal symbols and algorithmic procedures (Reys, 1984; Reys, Bestgen, Rybolt, & Wyatt, 1982; Rubenstein, 1985); therefore, students' knowledge of estimation is often viewed as a computational skill. Hiebert and Wearne (1986) however, propose an alternative view of students' knowledge about estimation. They argue that estimation is an intuitive skill based upon prior conceptual knowledge, thus, students can estimate sums, differences, etc. in the context of real world situations and when working with symbolic representations prior to learning computational algorithms. They further suggest that by estimating sums, differences, etc. prior to learning computational algorithms, students avoid misconceptions commonly associated with knowledge of formal symbols and procedures. Therefore, Hiebert and Wearne suggest that students' knowledge about estimation can serve as a basis for developing concepts underlying mathematical symbols and procedures.

Few studies have examined students' knowledge about estimation with respect to fractions, but those that have, have primarily shown students failing to accurately estimate sums when working with symbolic representations (Behr, Wachsmuth, & Post, 1985; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). Both RNP and NAEP results characterize students' knowledge of estimation by its misconceptions, which are similar to those commonly associated with adding and subtracting fractions (Behr, Lesh, Post, & Silver, 1983; Carpenter et al., 1981; Erlwanger, 1973; Kerslake, 1986).

Because these studies examined students' knowledge with respect to fraction symbols rather than as an intuitive skill, it is not clear if students possess intuitive knowledge about estimation with fractions or if this knowledge may serve as a basis for developing understanding of fraction symbols and procedures.

This paper focuses on students' intuitive knowledge about estimation with fractions. The data for this paper comes from the study reported in Mack (1988), whose purpose was to examine the development of students' understanding about fractions during instruction with respect to the ways in which students built upon prior knowledge of fraction ideas to give meaning to formal symbols and procedures. This paper examines student's prior knowledge about estimation and provides insights into the nature of this knowledge and the ways in which students are able to build upon this prior knowledge to give meaning to fraction symbols and computational algorithms.

METHODOLOGY

The methodology for this study emerged from three primary sources: (1) the case study (Erickson, 1986; Shulman, 1986), (2) instructional approaches that utilize instruction to influence cognitive changes (Carpenter, 1987; Hiebert & Wearne, in press), and (3) the use of students' verbal reports as data (Ginsburg, Kossan, Schwartz, & Swanson, 1983). The details of the methodology are reported in Mack (1988); therefore, only brief descriptions of each component of the methodology are discussed here.

Sample

The sample consisted of eight sixth-grade students of average mathematical ability, who were identified as having little understanding about fractions. The students were initially identified by their classroom teachers and were then interviewed to screen those who demonstrated a strong understanding about fractions. All subjects came from a middle school that predominantly draws students from middle to upper-middle income families in Madison, Wisconsin. Prior to, and during this study, none of the students received instruction on fractions in their sixth-grade mathematics class.

Initial Assessment

Instruction started with assessment. Each student's knowledge was assessed in what was referred to as the Initial Assessment. The Initial Assessment served two purposes: (1) assessing the student's prior knowledge with respect to topics related to the addition and subtraction of fractions, and (2) identifying the student's misconceptions. The student was asked questions involving characterizing fractions, estimating sums and differences involving fractions, identifying equivalent fractions, partitioning a unit, and adding and subtracting like and unlike fractions.

The Initial Assessment was conducted as a clinical interview; therefore, the student's thinking with respect to each question was probed in various ways. The nature of the probing was determined by the student's response to the question and answers to previous questions (Ginsburg et al., 1983). Not all of the students were asked the same questions during the Initial Assessment. Students were not asked questions for which their prerequisite knowledge appeared deficient.

Assessment Tasks

Each question the student was given was regarded as an assessment task. All of the tasks were based upon four central ideas that emerged from a rational task analysis for the addition and subtraction of fractions: (1) determining the relationship between the number of parts a unit is divided into and the size of the parts, (2) a fraction is a single number with a specific value, (3) different fractions represent the same amount, and (4) the addition and subtraction of fractions requires common denominators.

The specific tasks the student received were based upon the central ideas of the rational task analysis combined with the student's reponses to previous questions and his or her choice of context for the problems. The tasks were ones that encouraged the student to draw upon his or her prior knowledge and to form relationships between pieces of knowledge.

The tasks were used to provide direction for instruction as well as to assess the student's thinking. In general, in situations where the student was unable to successfully solve a problem due to a misconception or lack of knowledge about an idea related to the problem, the student was given a simpler problem. In situations where the student successfully solved the problem by relating pieces of knowledge but the relationship appeared to be tenous, the student continued with a similar task. In situations where the student successfully solved a problem by relating pieces of knowledge and the relationship appeared to be strong, the student was given a problem that was closely related but more complex.

General Characteristics of the Instructional Sessions

Each student was regarded as an independent case study and received instruction in a one-to-one instructional session (subject and author of this paper) from 11-13 times over a period of six weeks. All instructional sessions lasted 30 minutes and occurred during regular school hours.

All instructional sessions combined clinical interviews with instruction; therefore, instruction was not scripted. The majority of the problems were presented to the student verbally. The student was encouraged to think aloud as he or she solved problems. If the student failed to think aloud, the student was asked to explain what he or she had been thinking as the problems were solved.

The instructional content deviated from topics covered in chapters on fractions in traditional textbook series in two important ways: (1) the student's intuitive understanding about fractions provided the basis for instruction (Carpenter, 1987), and (2) the estimation of fractions was emphasized (Hiebert & Wearne, 1986; NCTM, 1980; Reys, 1984). Estimation was viewed as an intuitive skill (Hiebert & Wearne, 1986), and the specific situations in which instruction emphasized estimation consisted of three components: (1) examining individual fractions represented by concrete materials, real world situations, and symbolic representations and estimating the quantity represented, (2) estimating sums and differences involving fractions, and (3) constructing sums or differences that are close to, but not equal to one.

Concrete materials in the form of fraction circles and fraction strips were available for the student to use, and their use was encouraged as long as the student thought they were needed. Pencil and paper were available for the student's use; however, their use was not encouraged until the student had successfully solved problems using the concrete materials in situations where misconceptions initially appeared when using pencil and paper.

After the Initial Assessment and each instructional session, a lesson was planned for the student's next session that was based upon the student's prior knowledge, misconceptions, responses to problems presented in previous sessions, and relationships between components of the instructional content. The lessons were desinged to be flexible both prior to and during instruction to deal with the student's misconceptions and to provide variety and motivation for the student when he or she began to show signs of frustration or boredom. A rational task analysis for the addition and subtraction of fractions provided structure for the flexibility of the lessons.

All instructional sessions were audio-taped. Each day I wrote out detailed notes from the student's audio-taped session and transcribed critical protocol segments. The notes and protocols were used to aid in planning instruction for the following session and in the data analysis. The student's protocols were reviewed several times during the study and after the conclusion of the study to identify relationships the student had formed between pieces of knowledge. When relationships were identified, they were compared to pieces of knowledge that instruction had attempted to relate to determine if the student had related the same pieces of knowledge, and possibly if they had related pieces of knowledge that had not yet been related through instruction. Each time I reviewed the protocols I found the same relationships that had been identified in earlier analyses, plus a few more.

Individualizing Instruction

Throughout the study, the author reacted to individual students; therefore, the specific manner in which instruction built upon intuitive understanding varied. In general, the author continually assessed the student's thinking and adjusted instruction to make the problems that drew upon prior knowledge and new knowledge more and more similar.

RESULTS

Instruction was specifically designed to build upon the student's prior knowledge and to emphasize estimation; therefore, instruction may have influenced the results. This section integrates individual protocols into a discussion of specific findings. Although some of the real world situations appear unrealistic, the student chose the context for his or her problems, such as cakes, pies, boards, etc. at the beginning of each session. Situations in which students estimated sums and differences when the problems were represented with fraction symbols are referred to as symbolic estimation, and situations in which students estimated sums and differences when the problems were presented in the context of real world situations without fraction symbols are referred to as real world estimation.

All eight students came to instruction with a substantial store of prior knowledge about real world estimation; however, this knowledge was unrelated to knowledge of fraction symbols and procedures. The students' knowledge of real world estimation focused on the size of distinct parts of a whole rather than on a whole divided into equal-sized parts with a certain number of parts designated. Laura illustrated the general nature of this knowledge during her third instructional session when she drew a picture to represent "about 3/5 of a pizza". She drew approximately 3/5 of a circle without drawing the complete circle or partitioning the circle, and explained her drawing as "it's more than half a pizza".

During the Initial Assessment, all eight students accurately estimated sums and differences involving whole numbers or decimals when the problems involved symbolic estimation and/or real world estimation. They also explained estimation as "It's like rounding". However, all of the students failed to accurately estimate the sum of 7/8 and 5/6 when the problem was represented with fraction symbols. Their estimates varied from 12/14 to 20. When the problem was restated in the context of a real world situation, the students accurately estimated the answer by examining each fraction individually and stating that each fraction "is close to one". When questioned about the inconsistencies between estimates for the symbolic estimation and the real world estimation, some students were bothered by the inconsistencies and responded that the estimates should be the same. These students responded that the real world estimate was accurate because "I can see it". Other students however, were not bothered by the inconsistencies and explained their estimates in terms of numbers as being one thing and pizzas being another.

Ned and Aaron provide examples of the types of misconceptions students had for symbolic estimation and of the prior knowledge they had about real world estimation. They also provide examples of the different ways that students responded to the inconsistencies between their symbolic estimates and their real world estimates. The following protocols were taken from both Ned's and Aaron's Initial Assessments, which was both students first experience estimating sums and differences involving fractions.

<u>Ned</u>

I: (gave Ned piece of paper with 7/8 + 5/6 printed on it) If you estimate this answer, what would you get?

Ned: (wrote 10/10, but left out /, then wrote 20) Twenty.

I: If you had 7/8 of a pizza, do you have more than a pizza or less than a pizza?

Ned: Less than . . . It's alot of pizza but it's still less than a whole one (got out fraction circles on his own and showed 7/8 then 5/6), and this (5/6) is about the same as that (7/8), so you have about two pizzas. (paused, looked at paper) That's (2) not close to 20.
I: How did you get 20 for an estimate?
Ned: I added these and then I estimated it.
I: Which one do you think is right, two or 20?
Ned: Two . . . 'cause I can see it.

Aaron

I: (gave Aaron piece of paper with 7/8 + 5/6 printed on it) I want you to estimate the answer to this problem, 7/8 plus 5/6.
Aaron: Twelve-fourteenths.
I: Why do you think it's 12/14?
Aaron: Well, eight plus six is 14, and seven plus five is 12. . . It's close to one. . .
I: . . . If you had 7/8 of a cheese pizza and 5/6 of a sausage pizza, about how much pizza do you have?
Aaron: Eleven-fifteenths, no.
I: Let's see if we can figure it out. . . We've got 7/8 of a cheese pizza, about how much of a pizza is that?
Aaron: The whole thing except for one piece is gone. Oh, I see now! . . . Well, it's like two pizzas with only two pieces missing, one on each.
I: . . . You said this was close to one (referring to estimate of 12/14).
Aaron: Ohhh!
I: And when you estimated with pizzas this is close to two.
Aaron: Yea.
I: Well now why would you get one when you added, but when we used pizzas we get close to two?
Aaron: Well, it's kind of like one pizza, one pizza with two pieces missing, or it's two pizzas with one missing on each.

Ned's explanation for his estimate of 20 suggested that he was treating numerators and denominators as independent whole numbers and employing a rounding procedure he knew for estimating sums involving whole numbers. Aaron's estimate of 12/14 suggested that he was focusing on an incorrect procedure for adding fractions, adding numerators together and adding denominators together, rather than on estimating the sum. Both students' estimates of two for the problem restated in the context of a pizzas illustrated that they had prior knowledge about real world estimation that was initially unrelated to knowledge of fraction symbols; however, Ned's response that "two is not close to 20" suggested that he knew he should obtain the same answer for both problems. His explanation for two as an accurate estimate suggested that he had related the symbolic representation for the problem to his prior knowledge of real world estimation, while Aaron's explanation of "one pizza" and "two pizzas" suggested that he viewed fraction symbols and his prior knowledge of real world situations as being unrelated.

Other students responded in a manner similar to Ned's or Aaron's when asked to estimate the answer to 7/8 + 5/6. Some of them applied a combination of rounding and adding fractions and others applied incorrect procedures for adding fractions when working with symbolic estimation. All of the students accurately estimated the sum by examining the fractions independently and choosing references for them when the problem was restated in the context of a real world situation. Like Ned, some of them resolved the inconsistencies between their estimates by explaining the symbolic estimation in terms of real world estimation, and others, one student, viewed fraction symbols and real world situations as unrelated things.

After the Initial Assessment, students continued estimating sums and differences presented in the context of real world situations by selecting references for individual fractions on their own, such as 1/4, 1/2, 1 etc. The only difficulties students encountered when working with real world estimation occurred when they received problems involving fractions that were unfamiliar. Teresa illustrated these difficulties during her second instructional session when she used real world estimation for a problem involving 6/10 plus 1/25. She explained the problem as "Six-tenths is about 1/2, and 1/25 is about 1/10 or something like that . . . I know it's more than 1/100, but I don't really know what it is".

The students also attempted to select references for individual fractions on their own when working with symbolic estimation; however, all of them quickly encountered problems involving symbolic estimation in which they were uncertain of which references to select or of how to solve the problem after selecting a specific reference. For most of the students, this occurred during his or her third or fourth instructional session. Prior to this, all of the students demonstrated that they had prior knowledge about fraction ideas related to half of a whole and one whole and about joining and separating various combinations of these two. Therefore, when students initially experienced difficulties when working with symbolic estimation, instruction suggested that they begin by using 0, 1/2, and 1 as references, but they could use others if desired. Instruction was limited to suggesting these references and gave students latitudes in inventing ways for determining the appropriate reference.

Julie and Bob provide examples of the difficulties students had in selecting references for symbolic estimation and of the various ways students invented for determining appropriate references. The following protocols were taken from Julie's third instructional session and Bob's fifth and sixth session.

<u>Julie - Third Instructional Session</u>

I: I want you to estimate the answer to this problem, 9/10 minus 1/15.
Julie: (wrote 9/10 - 1/15 on her paper) This (9/10) is about 10/10... one whole.
I: And what about 1/15?
Julie: (pause)
I: Suppose you got 1/15 of a pizza, about how much would you get?
Julie: (pause) How do you say it's small?
I: If it's really small, I usually say it's zero.
Julie: (wrote 1 - 0 = 1 on her paper)
I: (explained can use 0, 1/2, and 1 as references) ... Okay, now estimate the answer to this problem, 3/8 plus 5/9...
Julie: ... [3/8] is about 1/2 because 4 plus 4 is 8 so 4/8 is half and 3/8 is one away ... [5/9], it's over a half... one piece... 4 1/2 is half of 9... it's half a piece over... so 1/2 plus 1/2 is one whole.

<u>Bob - Fifth Instructional Session</u>

I: Suppose you have 8/9 of a chocolate cake and I give you 3/7 more of a chocolate cake, about how much chocolate cake do you have?

Bob: Eight-ninths I think would be about four and one-half fifths... or 4/5... and 3/7 is about 1/2 'cause 3 1/2 is half of 7, so 4/5 plus 1/2 (pause). I don't know how to do that.

I: It might help if you tried to see if the fractions are closer to 0, 1/2, or one. You know how to add those, but you can use others if you want. The better we get at estimating, the more references we can use. (Time was up and another student was waiting so the session ended with this.)

Bob - Sixth Instructional Session

I: I want you to estimate the answer to this problem, 8/9 + 3/7.

Bob: Well, 8/9 is about two whole.

I: Are you sure?

Bob: No, it's about one whole because 9/9 is one whole, and 3/7 is about a half ... 3 1/2 is half of 7 and 3 is only half away from a half... so it's about 1 1/2...

I: ... Suppose you have about 2 2/3 cakes and you and your brother eat about 1 3/4 cakes, about how much cake do you have left?

Bob: Two-thirds is about a whole.

I: Is it closer to a whole than a half?

Bob: No, it's closer to a half, because 1 1/2 is half of 3, and 3/3 is a whole, and 2 away from 3/3 is one whole number, and those two away from 1/2 is only a half, so that'd be 2 1/2 (rewrote 2 2/3 as 2 1/2). Subtract one and 3/4 is like 2/3, wait, no it's not, wait 2/3... 3/4 is closer to, it's on the line between 2 and 4... [it's] almost 2/4 or almost 4/4... so it's equal... I'll go down (rewrote 1 3/4 as 1 2/4)... 2/4 is the same as a half, so I can write it 1 1/2, okay, so that makes it easier, then you get one whole.

Julie's response "How do you say it's small" suggested that she had a sense of the size of 1/15 but that she was uncertain of which reference to choose. Bob's selection of 4/5 as a reference for 8/9 illustrated that he had a sense of the size of 8/9 and that he was uncertain of how to solve the problem after selecting the reference. Both Julie's explanations for why 3/8 and 5/9 are close to 1/2 and Bob's explanations for why 3/7, 2/3, and 3/4 are close to 1/2 illustrated that they invented their own ways of determining appropriate references, comparing individual fractions to the number of parts needed to make half of a whole and the number of parts needed to make a whole.

Other students responded in a manner similar to Julie's and Bob's after instruction suggested 0, 1/2, and 1 as references for symbolic estimation. They invented similar ways for determining appropriate references, or as Tony explained when solving a problem involving 2/12,

> "[2/12] is about none of a board, because half of a board is 6/12 'cause 6 plus 6 is 12, and a whole board is 12/12, and zero of a board is 0/12, and this(2/12) is closer to zero than it is to 6 or 12".

During the Initial Assessment, seven students communicated misconceptions they had for adding and subtracting fractions, such as adding numerators together and adding denominators together. However, all of the students accurately estimated sums and differences when working with symbolic estimation prior to covering computational algorithms for adding and subtracting unlike fractions. When students did encounter problems requiring them to add and subtract unlike fractions, four of them, on their own initiative, extended knowledge about estimation to these problems in various ways. This knowledge about estimation aided them in determining how to add and subtract unlike fractions, dealing with misconceptions related to computational algorithms, and assessing the reasonableness of answers.

Ned, Bob, and Teresa provide examples of the various ways in which students extended knowledge of real world estimation to computational algorithms. The following protocols were taken from Ned's ninth instructional session, Bob's eighth session, and Teresa's ninth session. Prior to these sessions, Ned said that fractions needed to have the same "bottom number" before he could add them, but he was unsure of how to find the common denominators, Bob insisted that when the "bottom numbers" are different you add the top ones together and the bottom ones together", and Teresa was unsure of which denominator to change when adding and subtracting unlike fractions and frequently committed computational errors.

Ned - Ninth Instructional Session

I: I want you to solve this problem, 5/8 plus 1/2.
Ned: (wrote 5/8 + 1/2). This (5/8) is about close to a half isn't it?
I: Yes, it's real close to a half, so what do you think the answer's going to be?
Ned: About one, . . . well 5/8 is close to 4/8 and 4/8 is, four plus four is eight so if that (indicating 5/8) was a half [plus] a half, one whole.
I: That's a good estimate and now what is the exact answer? . . .
Ned: . . . (wrote 5/8 + 4/8 = 9/8) Oh, well you could make one whole and have two pieces left over.
I: Two pieces?
Ned: One piece. . . well 'cause 8/8 make a whole pie and you have 9/8.

Bob - Eighth Instructional Session

I: Suppose you have 1/4 of a chocolate cake and I give you 1/3 more of a chocolate cake, how much chocolate cake do you have?
Bob: I'll estimate it first because then I'll know if my answer is right. (pause) I'd say about, maybe a little over a half.
I: Why a little over a half?
Bob: Well, I took a little under a half I mean.
I: Why a little under a half?
Bob: Okay, I took 1 + 1 that'd be 2, and 3 + 3 that'd be 6.
I: Wait a minute, where's the 1 + 1 is 2 and 3 + 3 is 6? Where'd they come from?
Bob: There's the 1 + 1 (points to the numerators), and then I decided to change this (4 in 1/4) to a 3, so it's easier to add it. Well, so you can add it easily . . . I'll start over, 1/4 is 1/4, and 1/3 is a little more than 1/4, so you get more than a half ... not very much more, like and eighth more...

Teresa - Ninth Instructional Session

I: Try this problem, 1 1/2 plus 2 3/4.
Teresa: (pause, mumbling to herself) Think I'm getting confused.
I: What are you getting confused about? Tell me what you're thinking and then I can help you out.
Teresa: Well 2, 1/2 times two two's equals four two's or 2/4 and so (pause), but then (pause), you'd leave it the same 'cause it's the same denominator, and the four times two is five and that'd be an improper fraction.
I: Wait, four times two is five? Where'd you get four times two?
Teresa: I mean, I mean, wait three plus two is five.
I: Why did you add three plus two?
Teresa: Well, you have to since this (1/2) is gonna be 2. . . (wrote 4 5/4 for answer) Actually, this (4 5/4) is wrong. . . 'Cause it would be 3 5/4 'cause if you use common sense, this (1 1/2) is 1 1/2 and this (2 3/4) is about 2 1/2, so that'd give you 4, a little over 4, but this (4 5/4) is over 5 so that can't be right, so it's really 3 5/4, then you can reduce it to 4 1/4.

Ned's response that 5/8 is close to 1/2 suggested that he was using his knowledge about estimation to determine how to add 5/8 and 1/2. Bob's comment that he would estimate his answer first suggested that he was using his knowledge about estimation to deal with his misconception for adding unlike fractions, which was revealed by his response that 1/3 plus 1/3 equals 2/6. His response that 1/3 is a little more than 1/4 suggest that he had adopted additional references for estimation. Teresa's explanation for why she was confused suggested she was focusing on a procedure when adding the fractions; however, her response that her answer was incorrect suggested that she had utilized her knowledge about estimation to assess the reasonableness of her answer. All three students' explanations further illustrated that they had drawn on their knowledge about estimation on their own, and through their own efforts extended this knowledge to computational algorithms.

Ned, Bob, Teresa, and one other student, Aaron, frequently applied knowledge about estimation to problems involving adding and subtracting unlike fractions. They frequently utilized this knowledge to "estimate it first because then I'll know if my answer is right" or to check computations "to tell if my answer is right". In all of these situations, the students drew upon their prior knowledge on their own initiative. The other four students focused on prior knowledge about procedures for adding and subtracting fractions, which were often incorrect, when they encountered problems involving adding and subtracting unlike fractions. These students however, were able to accurately estimate sums and differences when asked to do so, but on their own initiative they did not extend knowledge about estimation to computational algorithms.

The four students who extended knowledge about estimation to computational algorithms became proficient at symbolic estimation by adopting additional references on their own. Common additional references were 1/4, 3/4, 1/8, 3/8, 5/8, and 7/8; however, the students continually adopted others. Bob and Teresa provide examples of the proficiency students attained for symbolic estimation by building upon prior knowledge of real world estimation. The following protocols were taken from Bob's eighth and twelfth instructional sessions and Teresa's eleventh session. The protocols illustrate both students proficiency with symbolic estimation as well as their various inventions for determining appropriate references and their flexibility in applying this knowledge.

Bob – Eighth Instructional Session

I: Pretend that you have 4/5 of a chocolate cake and I give you 9/10 more, how much chocolate cake do you have?

Bob: (wrote $4/5 + 9/10 \approx 1\ 2/3$) About 1 2/3, because 4/5 is 1/5 away from a whole and 9/10 is 1/10 away from a whole, so they're both about a whole, but then they're also one away, 1/5 and 1/10 away from a whole, so I thought of 2/3, 'cause a fifth and a tenth are about 2/3, I mean 1/3.

Bob – Twelfth Instructional Session

I: (gave Bob cards with numbers on them, one number on each card, numbers ranged from 1 – 12, also gave him a piece of paper with $\frac{-}{-} + \frac{-}{-} + \frac{-}{-} \approx 1$ written on it) I want you to use these numbers, you don't have to use all of them and you can't use any of them more than once, but you have to make three fractions so that when you add them together you get about one. You can get a little more than one or a little less than one, but you can't get exactly one.

Bob: (quickly formed $7/11 + 3/10 + 1/12 \approx 1$) Well I put, 7/11 is just about, 11ths are about the same as tenths, and 7 plus 3 is 10, and then this (denominators of 11 and 10) is 11ths, I just thought that up, and you have 10 (10/11), and so if you just add one more that'd be 11, and [12ths] are a little littler [than 11ths], so that'd be about one.

Teresa – Eleventh Instructional Session

I: (gave Teresa cards with one number on each of them, numbers were from 0 – 8, and gave her a piece of paper with $\frac{-}{-} + \frac{-}{-} \approx 1$ written on it) I want you to make two fractions that will add to be almost one, but you can't get exactly one...

Teresa: (wrote $3/4 + 5/6 \approx 1$) Oh this is closer to two, this [sum] is close to two 'cause this (3/4) is close to one, and this (5/6) is close to one, and one plus one is two. That's not going to work . . . How close? What's the farthest I can go away?

I: Well, I'm not going to tell you that, just get as close as you can.

Teresa: I'm gonna use common sense. I'm gonna say that one of them is gonna be pretty close to one, and then one of them has to be kinda far away.
I: Far away from what?
Teresa: Far away from, from one.
I: So what will it be close to?
Teresa: Maybe 1/8. I think one of 'ems 1/8, and then one of them is (pause) i think it's 5/6... But I also could have used two of 'em that were about the same as 1/2.

Bob's explanation that a fifth and a tenth are about a third and his formation of three addends whose sum is close to one illustrated his proficiency with symbolic estimation, as well as his flexibility in applying this prior knowledge to computational algorithms. His response of "Ijust thought that up" when explaining that tenths, elevenths, and twelfths are about the same suggested that he had invented yet another way of determining appropriate references. Teresa's response that one of the fractions could be close to one and the other close to zero, or that they could both be close to 1/2 illustrated not only her proficiency with symbolic estimation, but also her flexibility in applying her prior knowledge about estimation.

All of the students became better and better at symbolic estimation by building upon prior knowledge about real world estimation; however, four students became proficient at symbolic estimation by adopting additional references on their own. Whatever references the students utilized, they invented their own ways for determining appropriate references. Whether the students built upon knowledge of real world estimation to estimate sums and differences represented with fraction symbols prior to covering computational algorithms or to extend this knowledge to computational algorithms, they built upon this knowledge in similar ways, or as Teresa frequently remarked, "I'm gonna use common sense".

DISCUSSION

This investigation presents a different picture of students' knowledge about estimation with fractions than has been portrayed by previous studies. Whereas, previous studies showed students possessing numerous misconceptions for symbolic estimation, this investigation showed students possessing a rich store of prior knowledge about real world estimation that provided a basis for developing understanding about fraction symbols and computational algorithms.

The students' explanations in terms of one whole or half of a whole for their selection of references for individual fractions and their ability to estimate sums and differences prior to covering computational algorithms suggested that their prior knowledge about estimation focused on the size of distinct parts of a whole and was free of misconceptions commonly associated with knowledge of fraction symbols and procedures. Their explanations further suggested that they had a clear understanding about the size of various parts of a whole, such as 7/8 is "alot of pizza, about one whole" and 5/6 "is a little less of a pizza". However, the students' misconceptions when initially working with symbolic estimation suggested that their prior knowledge about estimation was unrelated to knowledge about fraction symbols and procedures. Therefore, this investigation suggests that students' prior knowledge about estimation is an intuitive skill emerging from a rich conceptual knowledge base.

The students' adoption of additional references, their alternative inventions for determining appropriate references, and their extensions of prior knowledge about estimation to computational algorithms suggested that not only did this prior knowledge serve as a basis for the development of their understanding about fraction symbols and procedures, but also that they built upon this knowledge in ways that were meaningful to them. Frequent responses such as "I'm gonna use common sense" or "I just thought that up" suggested that students were building upon their prior knowledge in meaningful ways. The students' flexibility in extending this prior knowledge to computational algorithms allowed then to add and subtract unlike fractions in situations where they had misconceptions related to computational algorithms, which further suggested that they were building upon this prior knowledge in meaningful ways. Whether inventing alternative ways for determining additional and appropriate references or flexibily extending knowledge about estimation to computational algorithms, the students' explanations illustrated that they drew upon prior knowledge about estimation on their own initiative. Therefore, this investigation suggests that as students built upon prior knowledge about estimation in meaningful ways, the students themselves assigned a critical role to this prior knowledge in the development of their understanding about fraction symbols and computational algorithms.

This investigation is a beginning in characterizing the nature of students' prior knowledge about estimation and the role that it may play in the development of students' understanding about fraction symbols and procedures. Although this investigation focused on students in an individualized instructional setting, similarities existed between the students' responses with respect to their selection of specific references, their alternative inventions for determining appropriate references, and their flexible use of prior knowledge about estimation. Therefore, this investigation suggests some general ways in which instruction can encourage students to build upon prior knowledge about estimation in regular classroom settings.

The student's responses illustrated that initially they were able to accurately estimate sums and differences when problems were presented in the context of real world situations but that they failed to accurately estimate the sums and differences when they were represented with fraction symbols. Therefore, to encourage students to build upon prior knowledge about estimation, instruction should initially present problems in the context of real world situations rather than presenting students with problems represented symbolically. For example, problems should be presented as "If you have about 7/8 of a pepperoni pizza and I give you about 5/6 of a cheese pizza, about how much pizza do you have?" prior to presenting problems such as "$7/8 + 5/6 \approx$?".

The students' responses further illustrated that even though they had prior knowledge about estimation, at times they experienced difficulties relating fraction symbols to their knowledge of real world estimation. Their responses also illustrated that after instruction suggested the use of 0, 1/2, and 1 as beginning references when working with symbolic estimation, the students successfully related fraction symbols to their prior knowledge and invented alternative ways for determining appropriate references. In building upon prior knowledge about estimation, students may require some assistance in relating fraction symbols to prior knowledge; therefore, instruction should suggest specific references for which students have demonstrated prior knowledge, such as 0, 1/2, and 1 as a beginning references for symbolic estimation. However, to allow students to build upon this prior knowledge in ways that are meaningful to them, students should be given latitudes in inventing alternative ways for determining appropriate references and they should be allowed to adopt references such as 2/3, 4/5, etc. if they so desire.

The students' ability to estimate sums and differences prior to covering computational algorithms and their decisions of when to use estimation suggested that they built upon prior knowledge about estimation to give meaning to computational algorithms. The students' explanation further illustrated that as they built upon prior knowledge to give meaning to computational algorithms, they avoided numerous misconceptions commonly associated with knowledge of fraction symbols and procedures. To encourage students to build upon prior knowledge about estimation, instruction should encourage students to estimate sums and differences prior to teaching them computational algorithms for fractions. Students should be given problems both in the context of real world situations, such as the example given above, and in the form of symbolic representations with instruction stressing that an approximate answer is acceptable, and at times all that is required. Additionally, students should be encouraged to use estimation to predict and/or assess the reasonableness of answers after learning computational algorithms. Therefore, instruction should give students latitudes in extending prior knowledge about estimation to computational algorithms.

Whatever the specific tasks are that students are given, they should be ones that draw upon prior knowledge of real world estimation rather than symbolic estimation to encourage students to build upon their prior knowledge about estimation. Furthermore, students should be allowed to build upon this prior knowledge in ways that are meaningful to them. In so doing, instruction may find that students do possess a rich store of prior knowledge about fraction ideas related to estimation that can serve as a basis for the development of their understanding about fraction symbols and algorithmic procedures.

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